

BLOCK (OR HAMILTONIAN) LIE SYMMETRY OF DISPERSIONLESS D TYPE DRINFELD-SOKOLOV HIERARCHY

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ABSTRACT. In this paper, the dispersionless D type Drinfeld-Sokolov hierarchy, i.e. a reduction of the dispersionless two-component BKP hierarchy, is studied. The additional symmetry flows of this hierarchy are presented. These flows form an infinite dimensional Lie algebra of Block type as well as a Lie algebra of Hamiltonian type.

PACS numbers: 02.20.Sv, 02.20.Tw, 02.30.Ik

Key words: Additional symmetry, Block Lie algebras, Hamiltonian Lie algebras, dispersionless Drinfeld-Sokolov hierarchy of type D, dispersionless two-component BKP hierarchy.

1. INTRODUCTION

Additional symmetry is an interesting topic in the study of integrable hierarchies which has been studied extensively in recent years. Additional symmetries of the Kadomtsev-Petviashvili (KP) hierarchy introduced by Orlov and Shulman [1] contain one kind of important symmetry, namely, the Virasoro symmetry, which is closely related to matrix model by means of the Virasoro constraint and string equation [2, 3, 4]. Two sub-hierarchies (BKP and CKP) of the KP hierarchy also possess the additional Virasoro symmetry [5, 6, 7, 8] with consideration of the reductions on the Lax operator. The 2-dimensional Toda Lattice (2dTL) hierarchy introduced by Ueno and Takasaki in [9] is natural to have the similar additional symmetry because of the similarity between the KP hierarchy and this hierarchy [4]. For the dispersionless Toda hierarchy [10, 11], additional symmetries can be used to give string equations and to solve Riemann-Hilbert problems.

Infinite dimensional Lie algebras of Block type, as generalizations of the well-known Virasoro algebra, have been studied intensively in literature (see, e.g., [12, 13, 14, 15]). In [16], we provide a Block type algebraic structure of the bigraded Toda hierarchy (BTH) [17, 18]. Later on, this Block type Lie algebra is found again in dispersionless bigraded Toda hierarchy [19]. We would like to mention that this Block type Lie algebra is in fact also a special case of Hamiltonian Lie algebras. As one of four families of the well-known infinite dimensional Lie algebras of Cartan type (see, e.g., [20, 21]), Hamiltonian Lie algebras, which usually possess additional structures of associative algebras such that they form Poisson algebras, appear naturally in Hamiltonian mechanics, and are also central in the study of quantum groups.

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For KP type integrable system, as a reduction of two-component BKP hierarchy [6, 22, 23, 24, 25, 26], the Drinfeld-Sokolov hierarchy of D type [23, 25, 27, 28] has similar reduced double dressing structures as BTH system and also has a Block (or Hamiltonian) type Lie symmetric structure [29]. A nature question is whether we can find Block (or Hamiltonian) type Lie symmetry in these two kinds of dispersionless KP type integrable system (the dispersionless two-component BKP hierarchy and dispersionless D type Drinfeld-Sokolov hierarchy). In this paper, the dispersionless D type Drinfeld-Sokolov hierarchy is proved to be a good model to derive Block (or Hamiltonian) type infinite dimensional Lie symmetry.

This paper is arranged as follows. As a reduction of the dispersionless two-component BKP hierarchy, the dispersionless D type Drinfeld-Sokolov hierarchy will be introduced in Section 2. The Block (or Hamiltonian) symmetries of the dispersionless D type Drinfeld-Sokolov hierarchy will be derived in Section 3.

2. DISPERSIONLESS D TYPE DRINFELD-SOKOLOV HIERARCHY

2.1. Dispersionless two-component BKP hierarchy. Underlying topological Landau-Ginzburg models of D-type, the dispersionless two-component BKP hierarchy was proposed in [22, 26]

$$\frac{\partial L}{\partial t_k} = \{(L^k)_+, L\}, \quad \frac{\partial L}{\partial \hat{t}_k} = \{-(\hat{L}^k)_-, L\}, \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (1)$$

$$\frac{\partial \hat{L}}{\partial t_k} = \{(L^k)_+, \hat{L}\}, \quad \frac{\partial \hat{L}}{\partial \hat{t}_k} = \{-(\hat{L}^k)_-, \hat{L}\}, \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (2)$$

where

$$L = p + \sum_{i=1}^{\infty} \bar{u}_i p^{1-2i}, \quad \hat{L} = \sum_{i=1}^{\infty} \hat{u}_i p^{2i-1}, \quad (3)$$

and the Poisson bracket $\{ , \}$ is defined by

$$\{f, g\} := \text{ad } f(g) = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p}. \quad (4)$$

It is easily to find the following antisymmetric condition

$$L(-p) = -L(p), \quad \hat{L}(-p) = -\hat{L}(p). \quad (5)$$

2.2. Dispersionless D type Drinfeld-Sokolov hierarchy. Assume a new Lax function \mathcal{L} which has the following relation with two Lax functions of the dispersionless two-component BKP hierarchy introduced in last subsection

$$\mathcal{L} = L^{2n} = \hat{L}^{-2}. \quad (6)$$

Then the Lax functions of dispersionless two-component BKP hierarchy will be reduced to the following Lax function of dispersionless D type Drinfeld-Sokolov hierarchy [23]

$$\mathcal{L} = p^{2n} + \sum_{i=1}^n u_i p^{2i-2} + \rho^2 p^{-2}. \quad (7)$$

One can easily find the Lax function \mathcal{L} of dispersionless D type Drinfeld–Sokolov hierarchy will satisfy the following symmetric condition

$$\mathcal{L}(-p) = \mathcal{L}(p). \quad (8)$$

The reduction eq. (6) inspires us to define two fractional operators as

$$\mathcal{L}^{\frac{1}{2n}} = p + \sum_{i \geq 1} \bar{u}_i p^{1-2i}, \quad \mathcal{L}^{\frac{1}{2}} = \hat{u}_{-1} p^{-1} + \sum_{i \geq 1} \hat{u}_i p^{2i-1}, \quad (9)$$

which satisfy

$$\mathcal{L}^{\frac{1}{2n}}(-p) = -\mathcal{L}^{\frac{1}{2n}}(p), \quad \mathcal{L}^{\frac{1}{2}}(-p) = -\mathcal{L}^{\frac{1}{2}}(p). \quad (10)$$

The dispersionless D type Drinfeld–Sokolov hierarchy being considered in this paper is defined by the following Lax equations:

$$\frac{\partial \mathcal{L}}{\partial t_k} = \{(\mathcal{L}^{\frac{k}{2n}})_+, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \hat{t}_k} = \{-(\mathcal{L}^{\frac{k}{2}})_-, \mathcal{L}\}, \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (11)$$

One can find the terms with p^0 in $\mathcal{L}^{\frac{k}{2n}}$ and $\mathcal{L}^{\frac{k}{2}}$ disappear.

This Lax function \mathcal{L} of dispersionless D type Drinfeld–Sokolov hierarchy has the following dressing structure

$$\mathcal{L} = e^{ad\varphi}(p^{2n}) = e^{ad\hat{\varphi}}(p^{-2}). \quad (12)$$

The two dressing functions have the following form

$$\varphi = \sum_{i=1}^{\infty} w_i p^{1-2i}, \quad \hat{\varphi} = \sum_{i=1}^{\infty} \hat{w}_i p^{2i-1}. \quad (13)$$

By tedious but standard computation, dispersionless Sato equations of dispersionless D type Drinfeld–Sokolov hierarchy can be derived in the following proposition.

Proposition 2.1. *\mathcal{L} is the Lax function of the dispersionless D type Drinfeld–Sokolov hierarchy if and only if there exists two Laurent series φ $\hat{\varphi}$ (dressing function) which satisfy the equations*

$$\nabla_{t_k, \varphi} \varphi = -(\mathcal{L}^{\frac{k}{2n}})_-, \quad \nabla_{t_k, \hat{\varphi}} \hat{\varphi} = (\mathcal{L}^{\frac{k}{2n}})_{\geq 1}, \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (14)$$

$$\nabla_{\hat{t}_k, \varphi} \varphi = -(\mathcal{L}^{\frac{k}{2}})_-, \quad \nabla_{\hat{t}_k, \hat{\varphi}} \hat{\varphi} = (\mathcal{L}^{\frac{k}{2}})_{\geq 1}, \quad k \in \mathbb{Z}_+^{\text{odd}}, \quad (15)$$

where $\nabla_{t, X} Y = \sum_{m=0}^{\infty} \frac{(adX)^m}{(m+1)!} \partial_t Y$.

In the next section, we shall show that this dispersionless D type Drinfeld–Sokolov hierarchies have a nice Block (or Hamiltonian) symmetry as its appearance in BTH [16].

3. BLOCK (OR HAMILTONIAN) SYMMETRIES OF DISPERSIONLESS D TYPE DRINFELD-SOKOLOV HIERARCHIES

3.1. Block (or Hamiltonian) symmetries. In this section, we will construct the flows of additional symmetry which form the well-known Block (or Hamiltonian) type infinite dimensional Lie algebra. Firstly we introduce dispersionless Orlov-Schulman operators as follows,

$$\mathcal{M} = e^{ad\varphi}(\Gamma), \quad \hat{\mathcal{M}} = e^{ad\hat{\varphi}}(\hat{\Gamma}),$$

where

$$\Gamma = \frac{xp^{1-2n}}{2n} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{k}{2n} t_k p^{k-2n}, \quad \hat{\Gamma} = -\frac{x}{2} p^3 - \frac{1}{2} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k \hat{t}_k p^{2-k}.$$

We can rewrite $\mathcal{M}, \hat{\mathcal{M}}$ in terms of \mathcal{L} as

$$\mathcal{M} = \frac{x\mathcal{L}^{\frac{1}{2n}-1}}{2n} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} v_k \mathcal{L}^{-\frac{k}{2n}-1} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{k}{2n} t_k \mathcal{L}^{\frac{k}{2n}-1}, \quad (16)$$

$$\hat{\mathcal{M}} = -\frac{x}{2} \mathcal{L}^{-\frac{3}{2}} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \bar{v}_k \mathcal{L}^{-\frac{k}{2}-1} - \frac{1}{2} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k \hat{t}_k \mathcal{L}^{\frac{k}{2}-1}. \quad (17)$$

It is easy to see the following lemma holds.

Lemma 3.1. *The operators \mathcal{M} and $\hat{\mathcal{M}}$ satisfy*

$$\{\mathcal{L}, \mathcal{M}\} = 1, \quad \{\mathcal{L}, \hat{\mathcal{M}}\} = 1; \quad (18)$$

and

$$\frac{\partial \bar{\mathcal{M}}}{\partial t_k} = \{(\mathcal{L}^{\frac{k}{2n}})_+, \bar{\mathcal{M}}\}, \quad \frac{\partial \bar{\mathcal{M}}}{\partial \hat{t}_k} = \{-(\mathcal{L}^{\frac{k}{2}})_-, \bar{\mathcal{M}}\}, \quad (19)$$

where $\bar{\mathcal{M}} = \mathcal{M}$ or $\hat{\mathcal{M}}, k \in \mathbb{Z}_+^{\text{odd}}$.

Proof The following calculation will lead to one part of the first equation of eq. (19)

$$\begin{aligned} \partial_{t_k} \mathcal{M} &= \partial_{t_k} e^{ad\varphi}(\Gamma) \\ &= e^{ad\varphi} \partial_{t_k}(\Gamma) + \{\nabla_{t_k, \varphi} \varphi, \mathcal{M}\} \\ &= e^{ad\varphi} \left(\frac{k}{2n} p^{k-2n} \right) + \{-(\mathcal{L}^{\frac{k}{2n}})_-, \mathcal{M}\} \\ &= \frac{k}{2n} \mathcal{L}^{\frac{k}{2n}-1} + \{-(\mathcal{L}^{\frac{k}{2n}})_-, \mathcal{M}\} \\ &= \{\mathcal{L}^{\frac{k}{2n}}, \mathcal{M}\} + \{-(\mathcal{L}^{\frac{k}{2n}})_-, \mathcal{M}\} \\ &= \{(\mathcal{L}^{\frac{k}{2n}})_+, \mathcal{M}\}. \end{aligned}$$

The other parts can be proved in similar ways. \square

We can formulate the following 2-form

$$\omega = dp \wedge dx + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} d(\mathcal{L}^{\frac{k}{2n}})_+ \wedge dt_k - \sum_{k \in \mathbb{Z}_+^{\text{odd}}} d(\mathcal{L}^{\frac{k}{2}})_- \wedge d\hat{t}_k \quad (20)$$

which satisfies

$$d\omega = 0, \quad \omega \wedge \omega = 0. \quad (21)$$

Further the following proposition can be derived.

Proposition 3.2. *The dispersionless D type Drinfeld–Sokolov hierarchy is equivalent to the following exterior differential equations.*

$$d\mathcal{L} \wedge d\mathcal{M} = d\mathcal{L} \wedge d\hat{\mathcal{M}} = \omega. \quad (22)$$

Proof The proof is standard and one can check the similar proofs in [11, 29]. \square

One can prove the following antisymmetric property of the dispersionless Orlov-Schulman operators \mathcal{M} and $\hat{\mathcal{M}}$,

$$\mathcal{M}(-p) = -\mathcal{M}(p), \quad \hat{\mathcal{M}}(-p) = -\hat{\mathcal{M}}(p). \quad (23)$$

For dispersionless D-type Drinfeld-Sokolov hierarchy, we define

$$\mathcal{B}_{m,l} = (\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l, \quad m \in \mathbb{Z}_+^{\text{odd}}, \quad (24)$$

and it is easy to check that

$$\mathcal{B}_{m,l}(-p) = -\mathcal{B}_{m,l}(p), \quad m \in \mathbb{Z}_+^{\text{odd}}. \quad (25)$$

That means it is reasonable to define additional flows of the dispersionless D type Drinfeld-Sokolov hierarchy as

$$\frac{\partial \mathcal{L}}{\partial c_{m,l}} = \{-(\mathcal{B}_{m,l})_-, \mathcal{L}\}, \quad m \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+. \quad (26)$$

The action of above additional flows on $\varphi, \hat{\varphi}$ should be as the following

$$\nabla_{c_{m,l}, \varphi} \varphi = -(\mathcal{B}_{m,l})_-, \quad \nabla_{c_{m,l}, \hat{\varphi}} \hat{\varphi} = (\mathcal{B}_{m,l})_+. \quad (27)$$

Further we can get the following identities hold

$$\frac{\partial \mathcal{M}}{\partial c_{m,l}} = \{-(\mathcal{B}_{m,l})_-, \mathcal{M}\}, \quad \frac{\partial \hat{\mathcal{M}}}{\partial c_{m,l}} = \{(\mathcal{B}_{m,l})_+, \hat{\mathcal{M}}\}, \quad m \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+. \quad (28)$$

The commutativity between different flows is a crucial property of integrable system. We shall show that additional flows defined by eq. (26) are commutative with flows of the dispersionless D type Drinfeld-Sokolov hierarchy (see the following proposition).

Proposition 3.3. *The additional flows in eq. (26) can commute with the original flow of dispersionless Drinfeld-Sokolov hierarchy of type D, namely,*

$$\left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial t_k} \right] = 0, \quad \left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial \hat{t}_k} \right] = 0, \quad m, k \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+$$

which hold in the sense of acting on \mathcal{L} .

Proof According to the definition, direct calculations can lead to

$$\begin{aligned} [\partial_{c_{m,l}}, \partial_{t_k}] \mathcal{L} &= \partial_{c_{m,l}}(\partial_{t_k} \mathcal{L}) - \partial_{t_k}(\partial_{c_{m,l}} \mathcal{L}), \\ &= -\partial_{c_{m,l}} \{(\mathcal{L}^{\frac{k}{2n}})_-, \mathcal{L}\} + \partial_{t_k} \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}\} \\ &= \{-(\partial_{c_{m,l}} \mathcal{L}^{\frac{k}{2n}})_-, \mathcal{L}\} - \{(\mathcal{L}^{\frac{k}{2n}})_-, (\partial_{c_{m,l}} \mathcal{L})\} \\ &\quad + \{[\partial_{t_k} ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)]_-, \mathcal{L}\} + \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, (\partial_{t_k} \mathcal{L})\}. \end{aligned}$$

Using eq. (11) and eq. (19), it equals

$$\begin{aligned} [\partial_{c_{m,l}}, \partial_{t_k}] \mathcal{L} &= \{ \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}^{\frac{k}{2n}}\}_-, \mathcal{L} \} + \{(\mathcal{L}^{\frac{k}{2n}})_-, \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}\}\} \\ &\quad + \{ \{(\mathcal{L}^{\frac{k}{2n}})_+, (\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l\}_-, \mathcal{L} \} - \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \{(\mathcal{L}^{\frac{k}{2n}})_-, \mathcal{L}\}\} \\ &= \{ \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}^{\frac{k}{2n}}\}_-, \mathcal{L} \} - \{ \{(\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l, (\mathcal{L}^{\frac{k}{2n}})_+\}_-, \mathcal{L} \} \\ &\quad + \{ \{(\mathcal{L}^{\frac{k}{2n}})_-, ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-\}_-, \mathcal{L} \} \\ &= 0. \end{aligned}$$

The other cases of this proposition can be proved in similar ways. \square

Proposition 3.3 indicates that eq. (26) is a symmetry of dispersionless D type Drinfeld-Sokolov hierarchy. Thus the flows in eq. (26) are called additional symmetry flows of the dispersionless D type Drinfeld-Sokolov hierarchy.

In order to have a better understanding of the properties of the additional symmetry flows, it is necessary to determine their algebraic structure. Using same techniques as in [19], the following theorem can be derived.

Theorem 3.4. *The flows in eq. (26) about additional symmetries of dispersionless D type Drinfeld-Sokolov hierarchy compose the following Block (or Hamiltonian) type Lie algebra*

$$[\partial_{c_{m,l}}, \partial_{c_{s,k}}] = (km - sl)\partial_{c_{m+s-1, k+l-1}}, \quad m, s \in \mathbb{Z}_+^{\text{odd}}, k, l \in \mathbb{Z}_+, \quad (29)$$

which holds in the sense of acting on \mathcal{L} .

Proof By using eq. (26) and eq. (28), we get

$$\begin{aligned} [\partial_{c_{m,l}}, \partial_{c_{s,k}}]\mathcal{L} &= \partial_{c_{m,l}}(\partial_{c_{s,k}}\mathcal{L}) - \partial_{c_{s,k}}(\partial_{c_{m,l}}\mathcal{L}) \\ &= -\partial_{c_{m,l}}\{((\mathcal{M} - \hat{\mathcal{M}})^s \mathcal{L}^k)_-, \mathcal{L}\} + \partial_{c_{s,k}}\{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}\} \\ &= -\{(\partial_{c_{m,l}}(\mathcal{M} - \hat{\mathcal{M}})^s \mathcal{L}^k)_-, \mathcal{L}\} - \{((\mathcal{M} - \hat{\mathcal{M}})^s \mathcal{L}^k)_-, (\partial_{c_{m,l}}\mathcal{L})\} \\ &\quad + \{(\partial_{c_{s,k}}(\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}\} + \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, (\partial_{c_{s,k}}\mathcal{L})\}, \end{aligned}$$

which further leads to

$$\begin{aligned} &[\partial_{c_{m,l}}, \partial_{c_{s,k}}]\mathcal{L} \\ &= -\left\{ \left[(\partial_{c_{m,l}}(\mathcal{M} - \hat{\mathcal{M}}))(\mathcal{M} - \hat{\mathcal{M}})^{s-1} \mathcal{L}^k + (\mathcal{M} - \hat{\mathcal{M}})^s (\partial_{c_{m,l}} \mathcal{L}^k) \right]_-, \mathcal{L} \right\} \\ &\quad - \{((\mathcal{M} - \hat{\mathcal{M}})^s \mathcal{L}^k)_-, (\partial_{c_{m,l}}\mathcal{L})\} \\ &\quad + \left\{ \left[(\partial_{c_{s,k}}(\mathcal{M} - \hat{\mathcal{M}}))(\mathcal{M} - \hat{\mathcal{M}})^{m-1} \mathcal{L}^l + (\mathcal{M} - \hat{\mathcal{M}})^m (\partial_{c_{s,k}} \mathcal{L}^l) \right]_-, \mathcal{L} \right\} \\ &\quad + \{((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, (\partial_{c_{s,k}}\mathcal{L})\} \\ &= \{(sl - km)(\mathcal{M} - \hat{\mathcal{M}})^{m+s-1} \mathcal{L}^{k+l-1}\}_-, \mathcal{L} \\ &= (km - sl)\partial_{c_{m+s-1, k+l-1}}\mathcal{L}. \end{aligned}$$

\square

As indices m, s in eq. (29) can only take values in odd numbers, the algebra is a subalgebra of the Lie algebra considered in [16], which is a Block type Lie algebra as well as a special case of Hamiltonian type Lie algebras (see, e.g., [20, 21]). The above results together with that on [19] shows that the Block (or Hamiltonian) type Lie algebras appear not only in dispersionless Toda type systems (dispersionless bigraded Toda hierarchy) but also in dispersionless KP type integrable systems (dispersionless D type Drinfeld-Sokolov hierarchy). These results also provide a strong support of the universality of Block (or Hamiltonian) type infinite dimensional Lie algebra in integrable hierarchies.

To see these Block (or Hamiltonian) symmetry flows more clearly, we give one example about a specific Block (or Hamiltonian) symmetry flow equation in the next subsection.

3.2. Example of Block (or Hamiltonian) symmetry flow. Here we consider the dispersionless D type Drinfeld-Sokolov hierarchy when $n = 1$, i.e. the hierarchy with Lax function as

$$\mathcal{L} = p^2 + u + \rho^2 p^{-2}. \quad (30)$$

As the first simplest nontrivial example, we shall discuss Block (or Hamiltonian) symmetry flow when $m = l = 1$ in eq. (26), i.e.

$$\frac{\partial \mathcal{L}}{\partial c_{1,1}} = \{-(\mathcal{B}_{1,1})_-, \mathcal{L}\} = \{-((\mathcal{M} - \hat{\mathcal{M}})\mathcal{L})_-, \mathcal{L}\}. \quad (31)$$

To avoid confusion, we denote different fractional functions $(\mathcal{L}^{\frac{k}{2n}}, \mathcal{L}^{\frac{k}{2}})$ when $n = 1$ as $(\mathcal{L}^{\frac{k}{2}}, \mathcal{L}^{\frac{k}{2}})$ respectively. From eq. (16) and eq. (17), one can easily get

$$((\mathcal{M} - \hat{\mathcal{M}})\mathcal{L})_- = \frac{x\mathcal{L}^{\frac{1}{2}}}{2} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} v_k \mathcal{L}^{-\frac{k}{2}} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{k}{2} t_k \mathcal{L}^{\frac{k}{2}} + \frac{1}{2} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k \hat{t}_k \mathcal{L}^{\frac{k}{2}}.$$

From dressing structure eq. (12) and eq. (16), the following relations between u, ρ and v_1, v_3 can be derived

$$u = -2\omega_{1x} = 4v_{1x}, \quad v_3 = -\frac{\omega_1 \omega_{1x}}{4} - \frac{3}{2}\omega_3, \quad \rho^2 = \omega_1 \omega_{1x} - 2\omega_3. \quad (32)$$

Suppose \mathcal{L} only depends on $x, t_1, \hat{t}_1, c_{1,1}$, then we can get

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{1,1}} &= \left\{ -\frac{(x+t_1)\mathcal{L}^{\frac{1}{2}}}{2} - v_1 \mathcal{L}^{-\frac{1}{2}} - v_3 \mathcal{L}^{-\frac{3}{2}} - \frac{1}{2} \hat{t}_1 \mathcal{L}^{\frac{1}{2}}, \mathcal{L} \right\} \\ &= \left\{ -\frac{(x+t_1)\mathcal{L}^{\frac{1}{2}}}{2} - v_1 \mathcal{L}^{-\frac{1}{2}} - v_3 \mathcal{L}^{-\frac{3}{2}} - \frac{1}{2} \hat{t}_1 \rho p^{-1}, \mathcal{L} \right\} \\ &= \frac{u}{2} + \frac{(x+t_1)}{2} \partial_{t_1} u + 2v_{1x} + \hat{t}_1 \rho_x \\ &\quad + \left(\frac{(x+t_1)}{2} \partial_{t_1} (\rho^2) + 2v_{3x} + \frac{1}{2} \hat{t}_1 \rho u_x \right) p^{-2}. \end{aligned} \quad (33)$$

which further leads to the following additional coupled flow equations over dynamic variables u, ρ

$$\frac{\partial u}{\partial c_{1,1}} = u + \frac{(x+t_1)}{2} \partial_{t_1} u + \hat{t}_1 \rho_x, \quad (34)$$

$$\frac{\partial \rho}{\partial c_{1,1}} = \frac{3\rho_x}{2} + \frac{(x+t_1)}{2} \partial_{t_1} \rho + \frac{1}{4} (u^2 + u_x \int u dx) \rho^{-1} + \frac{1}{4} \hat{t}_1 u_x. \quad (35)$$

Acknowledgments. Chuanzhong Li is supported by the National Natural Science Foundation of China under Grant No. 11201251, the Natural Science Foundation of Zhejiang Province under Grant No. LY12A01007 and the Natural Science Foundation of Ningbo under Grant No. 2013A610105. Jingsong He is supported by the National Natural Science Foundation of China under Grant No. 11271210 and K.C.Wong Magna Fund in Ningbo University. Yucai Su is supported by the National Science Foundation of China under Grant No. 11371278, the Shanghai Municipal Science and Technology Commission under Grant No. 12XD1405000 and the Fundamental Research Funds for the Central Universities of China.

REFERENCES

- [1] A. Yu. Orlov, E. I. Schulman, *Lett. Math. Phys.* **12** (1986) 171.
- [2] R. Dijkgraaf, E. Witten, *Nucl. Phys. B* **342** (1990) 486.
- [3] M. Douglas, *Phys. Lett. B* **238** (1990), 176.
- [4] M. Adler, T. Shiota, P. van Moerbeke, *Comm. Math. Phys.* **171** (1995), 547.
- [5] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, *Transformation groups for soliton equations. Nonlinear integrable systems-classical theory and quantum theory*, World Sci. Publishing, Singapore (1983).
- [6] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, *Phys. D* **4** (1981/82) 343.
- [7] K. Takasaki, *Lett. Math. Phys.* **28** (1993) 177.
- [8] J. S. He, K. L. Tian, A. Foerster and W. X. Ma, *Lett. Math. Phys.* **81** (2007) 119.
- [9] K. Ueno, K. Takasaki, *Adv. Studies in Pure Math.* **4** (1984) 1.
- [10] K. Takasaki, *Commun. Math. Phys.* **170** (1995) 101.
- [11] K. Takasaki, T. Takebe, *Rev. Math. Phys.* **7** (1995) 743.
- [12] R. Block, *Proc. Amer. Math. Soc.* **9** (1958) 613.
- [13] D. Dokovic, K. Zhao, *Algebra Colloq.* **3** (1996) 245.
- [14] Y. Su, *J. Algebra* **276** (2004) 117.
- [15] X. P. Xu, *Manuscripta Math.* **100** (1999) 489.
- [16] C. Z. Li, J. S. He, Y. C. Su, *J. Math. Phys.* **53** (2012) 013517.
- [17] C. Z. Li, J. S. He, K. Wu, Y. Cheng, *J. Math. Phys.* **51** (2010) 043514.
- [18] C. Z. Li, *J. Phys. A* **44** (2011), 255201.
- [19] C. Z. Li, J. S. He, *Rev. Math. Phys.* **24** (2012) 1230003.
- [20] V. G. Kac, *Math. USSR-Izvestija* **8** (1974) 801.
- [21] X. P. Xu, *J. Algebra* **224** (2000) 23.
- [22] K. Takasaki, *Lett. Math. Phys.*, **29** (1993) 111.
- [23] S. Q. Liu, C. Z. Wu, Y. Zhang, *Intern. Math. Res. Notices* **2011** (2011) 1952.
- [24] T. Shiota, *Infinite Dimensional Lie algebras and Groups* (1988) 407, *Adv. Ser. Math. Phys.* World Sci. Publ., Teaneck, NJ (1989).
- [25] C. Z. Wu, D. Xu, *J. Math. Phys.* **51** (2010), 063504.
- [26] Y. T. Chen, M. H. Tu, *J. Phys. A: Math. Gen.* **39** (2006) 7641.
- [27] V. G. Drinfeld, V. V. Sokolov, *J. Math. Sci.* **30** (1985) 1975.
- [28] C. Z. Wu, *Physica D* **249** (2013) 25.
- [29] C. Z. Li, J. S. He, *J. Math. Phys.* **54**(2013) 113501.